JOURNAL OF MULTIVARIATE ANALYSIS 31, 289–310 (1989)

A Matricial Extension of the Helson–Sarason Theorem and a Characterization of Some Multivariate Linearly Completely Regular Processes

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Communicated by the Editors

We generalize the theorems of Helson–Szegö and Helson–Sarason for matricial measures. We study two-weighted inequalities for the Hilbert transform in $[0, 2\pi]$ and in R and give a characterization for the positivity of the angle between past and future of multivariate weakly stationary stochastic processes, in the discrete and the continuous case. We also characterize the multivariate weakly stationary stochastic processes that are linearly completely regular and study the rate of convergence of the maximal correlation coefficient.

INTRODUCTION

The question concerning the positivity of the angle between past and future and the related problem concerning the complete regularity of a weakly stationary process were studied by several authors [9–13, 19] in the univariate case. The problem of the positivity of the angle between past and future is equivalent to that of the boundedness of the Hilbert transform, H, in the space $L^2(\mu)$, where μ is the spectral measure of the process and is answered by the theorem of Helson and Szegö. The positivity of the angle between the past and the future is equivalent to the boundedness of H in a corresponding subspace of $L^2(\mu)$ and is answered by a theorem of Helson and Sarason. In recent papers [16–18] the positivity of the angle between past and future was studied also for the multivariate process. In [17] Pourahmadi gave a matricial variant of the Helson–Szegö theorem for bounded spectral densities in $[0, 2\pi]$, which however does not include the full "exponential" formula of Helson and Szegö. In the present paper we extend the Pourahmadi result to a theorem of the Helson–Sarason form, for bounded matricial spectral measures μ in $[0, 2\pi]$ as well as in R.

Received September 29, 1988; revised April 18, 1989.

AMS 1980 subject classifications: Primary 60G25, 62M15; Secondary 42A50.

Key words and phrases: multivariate stationary process, completely regular process, prediction theory, Helson–Szegö theorem, Helson–Sarason theorem, weighted inequalities, Hilbert transform.

0047-259X/89 $3.00

The main aim of this paper is to give new conditions for μ to be the spectral measure of a process with positive angle between the past and the future, expressed in terms of the entries of the measure μ through a constructive exponential characterization of the Helson, Szegö, and Sarason types. From this result we characterize the multivariate linearly completely regular stationary processes (this was done in the scalar and discrete case in [9] and in the scalar and continuous case in [5]).

Let $\{r_n\}_{n \geqslant 0}$ and $\{r_t\}_{t \geqslant 0}$ be such that $r_n \downarrow 0$ as $n \to \infty$ and $r_t \downarrow 0$ as $t \to \infty$. Let $\rho_n(\mu)$ be the maximal correlation coefficient of a discrete process, in the scalar case Arocena, Cotlar, and Sadosky [4] proved that the theory of Weakly positive matrices can be used to obtain the theorems of Helson and Szegö and of Helson and Sarason with some refinements and gave some necessary conditions and other sufficient conditions for $\rho_n(\mu) = O(r_n)$. In [13] Ibragimov gave some sufficient conditions for $\rho_t(\mu) = O(t - {}^{(k+\alpha)})$ where $\rho_t(\mu)$ is the maximal correlation coefficient of a continuous process, α is a real number, k is a positive integer, and the spectral density is expressed in terms of an analytic function of exponential type and a function that has a kth derivative which satisfies a Hölder's type condition with exponent α; and in [5] we announced some necessary and sufficient conditions for $\rho_t(\mu) = O(r_t)$, answering in this way a problem stated by Ibragimov in a private communication. We also study the rate of convergence of the maximal correlation coefficient for multivariate processes, giving necessary and sufficient conditions for $\rho_n(\mu) = O(r_n)$ and for $\rho_t(\mu) = O(r_t)$.

We also give two weighted inequality results for the Hilbert transform in $[0, 2\pi]$ as well as in R (generalizing the results announced in [6, 7] for the scalar case). Let us remark that in the case of R we deal with spaces differently than with those considered by Adams [1] or by Arocena, Cotlar, and Sadosky [4]. In fact they consider functions with vanishing moments, while here as in [5–7], we study functions whose Fourier transform vanishes in an interval and obtain a characterization of the Helson–Szegö–Sarason type.

1. Weakly Positive Matrices of Measures with Matricial Values

Let $C_{q \times q} = \{\text{hermitian } q \times q \text{ matrices}\}$. $C_{q \times q}$ is a Hilbert space with scalar product $\langle\!\langle A, B \rangle\!\rangle_q = \operatorname{tr}(AB^*) = \sum_{i;k=1}^{q} a_{ik} \overline{b_{ik}}$, where $A, B \in C_{q \times q}$, $A = (a_{ik})$ and $B = (b_{ik})$. We extend to $C_{q \times q}$ some definitions used in [4] for $C_{1 \times 1}$. This can also be seen as a particular case of the definitions given in [3] for operator valued measures.

Let $H^1_{q\times q}([0, 2\pi]) = \{\text{matrices } (h_{ik});\ \text{where } h_{ik} \in H^1([0, 2\pi])\}$ and $M_{q\times q}([0, 2\pi]) = \{C_{q\times q}\text{-valued finite Borel measures } \mu \text{ in } [0, 2\pi]\}$.

Let $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ an hermitian matrix, with $\mu_{\alpha\beta} \in M_{q\times q}([0, 2\pi])$ for $\alpha, \beta = 1, 2$. We say that:

$(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ is positive if for all $\Delta \subset [0, 2\pi]$, for all $A_1, A_2 \in C_{q\times q}$

$$\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \langle\!\langle A_\alpha \cdot \mu_{\alpha\beta}(\Delta); A_\beta \rangle\!\rangle_q \geq 0.$$

$(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ is weakly positive if there exists $H \in H^1_{q\times q}([0, 2\pi])$ such that $(\mu_{\alpha\beta})_{\alpha,\beta=1,2} + \left[\begin{smallmatrix} 0 & H dx \\ H^* dx & 0 \end{smallmatrix}\right]$ is positive.

PROPOSITION 1.1. *Let $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ be an hermitian matrix with $\mu_{\alpha\beta} \in M_{q\times q}([0, 2\pi])$,*

$$\mu_{\alpha\beta} = \begin{bmatrix} (\mu_{\alpha\beta})_{11} & \cdots & (\mu_{\alpha\beta})_{1q} \\ \vdots & \begin{matrix}\cdots\\ \cdots\\ \cdots\end{matrix} & \vdots \\ (\mu_{\alpha\beta})_{q1} & \cdots & (\mu_{\alpha\beta})_{qq} \end{bmatrix} \qquad \textit{for} \quad \alpha, \beta = 1, 2.$$

Let

$$((\mu_{\alpha\beta})_{k_\alpha k_\beta})_{\alpha,\beta=1,2} = \begin{bmatrix} (\mu_{11})_{k_1 k_1} & (\mu_{12})_{k_1 k_2} \\ (\mu_{21})_{k_2 k_1} & (\mu_{22})_{k_2 k_2} \end{bmatrix}.$$

Then

(a) *$(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ is positive if and only if for all $k_1, k_2 \in \{1, ..., q\}$, $((\mu_{\alpha\beta})_{k_\alpha k_\beta})_{\alpha,\beta=1,2}$ is positive.*

(b) *$(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ is weakly positive if and only if for all $k_1, k_2 \in \{1, ..., q\}$, $((\mu_{\alpha\beta})_{k_\alpha k_\beta})_{\alpha,\beta=1,2}$ is weakly positive.*

Proof. (a) $(\leftarrow)$ Let $A_1, A_2 \in C_{q\times q}$ with $A_1 = (a^1_{ik})$ and $A_2 = (a^2_{ik})$, for $i_1, k_1, i_2, k_2 \in \{1, ..., q\}$ let $A^1_{i_1 k_1}$ and $A^2_{i_2 k_2}$ be the matrices given by $(A^1_{i_1 k_1})_{mn} = a^1_{i_1 k_1} \delta_{m i_1} \delta_{n k_1}$ and $(A^2_{i_2 k_2})_{mn} = a^2_{i_2 k_2} \delta_{m i_2} \delta_{n k_2}$, then

$$\begin{aligned} &\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \langle\!\langle A^\alpha_{i_\alpha k_\alpha} \cdot \mu_{\alpha\beta}(\Delta); A^\beta_{i_\beta k_\beta} \rangle\!\rangle_q \\ &\quad = \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \operatorname{tr}(A^\alpha_{i_\alpha k_\alpha} \cdot \mu_{\alpha\beta}(\Delta) \cdot A^\beta_{i_\beta k_\beta}{}^*) \\ &\quad = \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} a^\alpha_{i_\alpha k_\alpha} \overline{a^\beta_{i_\beta k_\beta}} (\mu_{\alpha\beta})_{k_\alpha k_\beta}(\Delta)\, \delta_{i_\alpha i_\beta} \end{aligned}$$

and, since $A_\alpha = \sum_{i_\alpha=1}^{q} \sum_{k_\alpha=1}^{q} A^\alpha_{i_\alpha k_\alpha}$ for $\alpha = 1,2$, the result follows.

$(\rightarrow)$ Let $\lambda_1, \lambda_2 \in C$ and $k_1, k_2 \in \{1, ..., q\}$ and $(A^{\alpha}_{k_\alpha})_{mn} = \lambda_\alpha \delta_{m1} \delta_{nk_\alpha}$ for $\alpha = 1, 2$. Then

$$\begin{aligned}\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} & \langle\!\langle A^{\alpha}_{k_\alpha} \cdot \mu_{\alpha\beta}(\Delta); A^{\beta}_{k_\beta} \rangle\!\rangle_q \\ &= \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \operatorname{tr}(A^{\alpha}_{k_\alpha} \cdot \mu_{\alpha\beta}(\Delta) \cdot A^{\beta *}_{k_\beta}) \\ &= \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \lambda_\alpha \overline{\lambda_\beta} (\mu_{\alpha\beta})_{k_\alpha k_\beta}(\Delta).\end{aligned}$$

(b) Follows immediately from (a).

PROPOSITION 1.2. *Let $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ be an hermitian matrix with $\mu_{\alpha\beta} \in M_{q\times q}([0, 2\pi])$ for $\alpha, \beta = 1, 2$, and $(\mu^{\#}_{\alpha\beta})_{\alpha,\beta=1,2}$ an hermitian matrix with $\mu^{\#}_{\alpha\beta} \in M_{1\times 1}([0, 2\pi])$ for $\alpha, \beta = 1, 2$. Let I_q be the identity matrix in $C_{q\times q}$. If there exist $0 < c < d < \infty$ such that $c \cdot \mu^{\#}_{\alpha\beta} \cdot I_q \leqslant \mu_{\alpha\beta} \leqslant d \cdot \mu^{\#}_{\alpha\beta} \cdot I_q$ for all $\alpha, \beta = 1, 2$ then:*

(a) *$(\mu^{\#}_{\alpha\beta})_{\alpha,\beta=1,2}$ is positive if and only if $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ is positive.*

(b) *$(\mu^{\#}_{\alpha\beta})_{\alpha,\beta=1,2}$ is weakly positive iff $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ is weakly positive.*

Proof. (a) $(\rightarrow)$ Let $k_1, k_2 \in \{1, ..., q\}$, and $\lambda_1, \lambda_2 \in C$. For all $A_1, A_2 \in C_{q\times q}$,

$$\begin{aligned}\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} & \langle\!\langle A_\alpha \cdot c \cdot \mu^{\#}_{\alpha\beta}(\Delta) \cdot I_q; A_\beta \rangle\!\rangle_q \\ &\leqslant \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \langle\!\langle A_\alpha \cdot \mu_{\alpha\beta}(\Delta); A_\beta \rangle\!\rangle_q.\end{aligned}$$

In particular, for $(A^{\alpha}_{k_\alpha})_{mn} = \lambda_\alpha \delta_{m1} \delta_{nk_\alpha}$ with $\alpha = 1, 2$ we have

$$\begin{aligned} c \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} & \lambda_\alpha \overline{\lambda_\beta} \mu^{\#}_{\alpha\beta}(\Delta) \cdot \delta_{k_\alpha k_\beta} \\ &\leqslant \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \lambda_\alpha \overline{\lambda_\beta} (\mu_{\alpha\beta})_{k_\alpha k_\beta}(\Delta).\end{aligned}$$

$(\leftarrow)$ If $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ is positive then, for $k = 1, ..., q$ $((\mu_{\alpha\beta})_{kk})_{\alpha,\beta=1,2}$ is positive, and so $(\mu^{\#}_{\alpha\beta})_{\alpha,\beta=1,2}$ is positive.

(b) $(\rightarrow)$ Suppose there is $h \in H^1([0, 2\pi])$ such that $(\mu^{\#}_{\alpha\beta})_{\alpha,\beta=1,2} +$

$\left[\begin{smallmatrix} 0 & h\,dx \\ \bar{h}\,dx & 0 \end{smallmatrix}\right]$ is positive. Let $h_{kj} = h \cdot \delta_{kj}$ for every $k, j \in \{1, ..., q\}$ and $H = (h_{kj})$ then

$$c \cdot (\mu_{12}^{\#} + h\,dx) \cdot I_q \leqslant \mu_{12} + c \cdot H\,dx,$$

$$c \cdot (\mu_{21}^{\#} + \bar{h}\,dx) \cdot I_q \leqslant \mu_{21} + c \cdot H^*\,dx.$$

Therefore $(\mu_{\alpha\beta})_{\alpha,\beta=1,2} + \left[\begin{smallmatrix} 0 & c \cdot H\,dx \\ c \cdot H^*\,dx & 0 \end{smallmatrix}\right]$ is positive.

($\leftarrow$) The converse part of (b) is similar to the converse part of (a). In this paper we shall use the following notations:

$$e_n \colon [0, 2\pi] \to C, \qquad e_n(x) = e^{inx},$$

$$P(C_{q\times q}) = \begin{cases} \phi \colon [0, 2\pi] \to C_{q\times q}; & \phi(x) = \sum_n e_n(x)\,\hat{\phi}(n) \\ \hat{\phi} \colon Z \to C_{q\times q} & \text{is a sequence of finite support} \end{cases}$$

$$P_1(C_{q\times q}) = \{\phi \in P(C_{q\times q});\ \hat{\phi}(n) = 0 \text{ for all } n < 0\},$$

$$P_2(C_{q\times q}) = \{\phi \in P(C_{q\times q});\ \hat{\phi}(n) = 0 \text{ for all } n \geqslant 0\}$$

$$P_{(n)}(C_{q\times q}) = \{\phi = e_n\phi_1 + \phi_2 \text{ for } \phi_1 \in P_1(C_{q\times q});\ \phi_2 \in P_2(C_{q\times q})\},$$
$$\text{for } n \geqslant 0.$$

Let $\mu \in M_{q\times q}([0, 2\pi])$ be fixed, we define

$$L_q^2([0, 2\pi], \mu) = \left\{\phi \colon [0; 2\pi] \to C_{q\times q} \text{ such that } \int_0^{2\pi} \phi(t)\,d\mu(t)\,\phi^*(t) \text{ exists}\right\},$$

where (using Rosenberg's definition, see [21]):

$$\left(\int_0^{2\pi} \phi\,d\mu\,\psi^*\right)_{i1} = \int_0^{2\pi} \left(\phi\,\frac{d\mu}{d\,\mathrm{tr}\,\mu}\,\psi^*\right)_{i1} d\,\mathrm{tr}\,\mu \qquad \text{and} \quad \mathrm{tr}\,\mu = \mathrm{trace}(\mu).$$

By a theorem of Rosenberg and Rozanov, $L_q^2([0, 2\pi], \mu)$ is complete under the inner product $\langle\!\langle \phi, \psi \rangle\!\rangle_\mu = \mathrm{tr} \int_0^{2\pi} \phi\,d\mu\,\psi^*$ (cf. [21]).

Let $\mu \in M_{q\times q}([0, 2\pi])$ then $P(C_{q\times q}) \subset L_q^2([0, 2\pi], \mu)$.

Let $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ be an hermitian matrix with $\mu_{\alpha\beta} \in M_{q\times q}([0, 2\pi])$ for $\alpha, \beta = 1, 2$. Let

$$M_q(\phi_1; \phi_2) = \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \langle\!\langle \phi_1; \phi_2 \rangle\!\rangle_{\mu_{\alpha\beta}}$$
$$\text{for} \quad (\phi_1, \phi_2) \in P(C_{q\times q}) \times P(C_{q\times q}).$$

PROPOSITION 1.3. *Let* $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ *be an hermitian matrix with* $\mu_{\alpha\beta} \in M_{q\times q}([0, 2\pi])$ *for* $\alpha, \beta = 1, 2$. *Then*:

(a) $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ *is positive if and only if*

$$M_q(\phi_1;\phi_2)\geqslant 0 \qquad \textit{for all} \quad (\phi_1,\phi_2)\in P(C_{q\times q})\times P(C_{q\times q}).$$

(b) $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ *is weakly positive if and only if*

$$M_q(\phi_1;\phi_2)\geqslant 0 \qquad \textit{for all} \quad (\phi_1,\phi_2)\in P_1(C_{q\times q})\times P_2(C_{q\times q}).$$

Proof. Use that this is known for the unidimensional case (see [4]).

2. Extension of the Helson-Sarason Theorem in the Discrete Case

Let $\mu, \nu\in M_{q\times q}([0,2\pi])$ be positive. Let **H** be the Hilbert transform defined linearly in $P(C_{q\times q})$ by

$$\mathbf{H}(\phi)=-i\phi \text{ if } \phi\in P_1(C_{q\times q}) \qquad \text{and} \qquad \mathbf{H}(\phi)=+i\phi \text{ if } \phi\in P_2(C_{q\times q}).$$

So that if $\phi\in P_{(n)}(C_{q\times q})$, $\phi=e_n\phi_1+\phi_2$ with $\phi_1\in P_1(C_{q\times q})$ and $\phi_2\in P_2(C_{q\times q})$ then $\mathbf{H}(\phi)=-ie_n\phi_1+i\phi_2$.

For $a\geqslant 1$, let

$$\begin{aligned} R_{q\times q}([0,2\pi],n,a)=\{(\mu;\nu)\in M_{q\times q}([0,2\pi])\times M_{q\times q}([0,2\pi]);\mu\geqslant 0;\\ \nu\geqslant 0 \text{ such that for all } \phi\in P_{(n)}(C_{q\times q})\\ \operatorname{tr}\textstyle\int_0^{2\pi}(H\phi)\,d\mu(H\phi)^*\leqslant a\cdot\operatorname{tr}\int_0^{2\pi}\phi\,d\nu\,\phi^*\}.\end{aligned}$$

We consider the problem of characterizing the elements of $R_{q\times q}([0,2\pi],n,a)$.

This can be reformulated as a prediction theory problem. Let (Ω,A,P) be a probability space. Let $(X_n)_{n\in Z}$ be a q-variated weakly stationary process, that is, $X_n=(X_{n(1)},...,X_{n(q)})$, $X_{n(j)}\in L^2(\Omega,A,P)$ for $j=1,...,q$ and the covariance depends only on $n-m$: $\Gamma_{n-m}=(E(X_{n(i)}\overline{X_{m(j)}}))_{i,j=1,...,q}$. The Herglotz theorem (see [15]) assures the existence of the spectral measure of the process $\mu\in M_{q\times q}([0,2\pi])$, which is positive hermitian. The maximal correlation coefficient of the process $(X_n)_{n\in Z}$ is:

$$\rho_n(\mu)=\sup_{\substack{\phi_1\in P_1(C_{q\times q});\,\phi_2\in P_2(C_{q\times q})\\ \langle\langle\phi_1;\phi_1\rangle\rangle_\mu=\langle\langle\phi_2;\phi_2\rangle\rangle_\mu=1}}|\langle\langle e_n\phi_1;\phi_2\rangle\rangle_\mu|.$$

The problem is to characterize the $\mu\in M_{q\times q}([0,2\pi])$ such that $\rho_n(\mu)\leqslant r$ for some $r\in(0,1)$.

We can define this coefficient $\rho_n(\mu)$ for any $\mu\in M_{q\times q}([0,2\pi])$.

PROPOSITION 2.1. *Let $a>1$, $n\geqslant 0$ and μ, $\nu\in M_{q\times q}([0,2\pi])$, $\mu\geqslant 0$, $\nu\geqslant 0$, and for α, $\beta=1,2$ let*

$$\mu_{\alpha\beta}=\frac{e_{n(\beta-\alpha)}(a\nu+(-1)^{|\beta-\alpha|+1}\mu)}{2\sqrt{a}}.$$

The following properties are equivalent:

(a) $(\mu,\nu)\in R_{q\times q}([0,2\pi],n,a)$.

(b) $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ *is weakly positive.*

If $\mu=\nu$ and

$$r=\frac{a-1}{a+1}\ \textit{then}\ \mu_{\alpha\beta}=\frac{e_{n(\beta-\alpha)}r^{1-|\beta-\alpha|}\mu}{\sqrt{1-r^2}}\qquad \textit{for}\quad \alpha,\beta=1,2$$

and conditions (a) *and* (b) *are also equivalent to*

(c) $\rho_n(\mu)\leqslant r$.

In this case μ is absolutely continuous.

Proof. The equivalence between (a), (b), and (c) is proved as in the unidimensional case (see [4]). Observe that if $\mu=\nu$ and $r=(a-1)/(a+1)$ then

$$a=\frac{1+r}{1-r},\qquad \frac{a-1}{2\sqrt{a}}=\frac{r}{\sqrt{1-r^2}},\qquad \frac{a+1}{2\sqrt{a}}=\frac{1}{\sqrt{1-r^2}}.$$

If $\mu=\nu$ then for every k, $j\in\{1,...,q\}$, $\left[\begin{smallmatrix}(\mu_{11})_{kk} & (\mu_{12})_{kj}\\ (\mu_{21})_{jk} & (\mu_{22})_{jj}\end{smallmatrix}\right]$ is weakly positive, so by the lifting theorem (see [4]) there is $h_{kj}\in H^1([0,2\pi])$ such that for all $\Delta\subset[0,2\pi]$,

$$\left|(a+1)\,e_n\mu_{kj}(\Delta)+\int_\Delta h_{kj}(x)\,dx\right|^2\leqslant (a-1)^2\,\mu_{kk}(\Delta)\,\mu_{jj}(\Delta).$$

If Δ is a set of Lebesgue measure zero, then

$$|(a+1)\,\mu_{kk}(\Delta)|^2\leqslant(a-1)^2\,\mu_{kk}(\Delta)\,\mu_{kk}(\Delta),$$

so $\mu_{kk}(\Delta)\leqslant-\mu_{kk}(\Delta)$ and $\mu_{kk}(\Delta)=0$. Thus $\mu_{kj}(\Delta)=0$. We finish the proof.

We shall say that g is a Helson–Sarason function of type n if there exist real bounded functions u and v with $\|v\|_\infty<\pi/2$ and an analytic polynomial P of degree $\leqslant n$, such that $g(x)=|P(x)|^2\exp(u(x)+\tilde{v}(x))$ (see [9]).

If we let $g(x)=|h(x)|\,e^{u(x)}$, where $h\in H^1([0,2\pi])$ and $v(x)=-\arg(e_{-n}(x)h(x))$ with $\|v\|_\infty<\pi/2$, then g is a Helson–Sarason function of type n (see [4]).

Let $V(r) = \text{arc}\sin r < \pi/2$ and

$$U(r, v) = \text{arc}\cosh\left(\frac{\cos v}{\sqrt{1-r^2}}\right) \qquad \text{for} \quad |v| \leqslant V(r).$$

PROPOSITION 2.2. *Let* $(z_{\alpha\beta})_{\alpha,\beta=1,2}$, *where* $z_{\alpha\beta} \in L^1([0, 2\pi])$ *such that*:

(i) $z_{11}(x) > 0$ *a.e.*, $z_{22}(x) > 0$ *a.e. and* $z_{12}(x) = \overline{z_{21}(x)}$ *a.e.*

(ii) $0 < \sqrt{z_{11}(x)\, z_{22}(x)} < |z_{12}(x)|$ *a.e.*

Let n be a nonnegative integer. We shall call $z(x) = z_{12}(x)\, e_{-n}(x)$ and $q = z_{12}/\sqrt{z_{11} z_{22}}$. The following conditions are equivalent:

(a) $Z = (z_{\alpha\beta}(x)\, dx)_{\alpha,\beta=1,2}$ is weakly positive,

(b) There exist a function $g(x) = |h(x)|\, e^{u(x)}$, where $h \in H^1([0, 2\pi])$ such that

$$\frac{|z_{12}(x)|}{|q(x)|} = \sqrt{z_{11}(x)\, z_{22}(x)} = \frac{1}{\sqrt{|q(x)|^2 - 1}}\, g(x),$$

and for $v(x) = -\arg(e_{-n}(x) h(x)\, e^{-ix \cdot \arg z(x)})$,

$$|u(x)| \leqslant U\left(\frac{1}{|q(x)|}; v(x)\right) \qquad \text{and} \qquad |v(x)| \leqslant V\left(\frac{1}{|q(x)|}\right) < \frac{\pi}{2}.$$

If there exist N such that $z(x) = |z(x)|\, e_N(x)$ then g is a Helson–Sarason function of type $n + N$.

Proof. If Z is weakly positive, then from the lifting theorem (see [4]) there exist $h \in H^1([0, 2\pi])$ and $\phi(x) = e_{-n}(x) h(x) e^{-ix \cdot \arg z(x)}$ such that

$$\begin{aligned} 0 &\geqslant |z_{12}|^2 - 2\,\mathrm{Re}(h\overline{z_{12}}) + |h|^2 - z_{11} z_{22} \\ &= (|q|^2 - 1)\, z_{11} z_{22} - 2|q|(\mathrm{Re}\, \phi)\sqrt{z_{11} z_{22}} + |\phi|^2. \end{aligned}$$

Let $v = -\arg \phi$, $u(x) = \log(\sqrt{z_{11} z_{22}}\sqrt{|q|^2 - 1}/|\phi|)$ and

$$z_k = \frac{|\phi|}{\sqrt{|q|^2 - 1}} \exp\left((-1)^k \,\text{arc}\cosh\left(\frac{|q| \cos v}{\sqrt{|q|^2 - 1}}\right)\right) \qquad \text{for} \quad k = 1, 2$$

then $z_1 \leqslant \sqrt{z_{11} z_{22}} \leqslant z_2$; that is,

$$|u(x)| = \left|\log\left(\frac{\sqrt{z_{11} z_{22}}\sqrt{|q|^2 - 1}}{|\phi|}\right)\right| \leqslant \text{arc}\cosh\left(\frac{|q| \cos v(x)}{\sqrt{|q|^2 - 1}}\right).$$

And $|\,|z_{12}(x)|\, e_n(x)\, e^{ix \cdot \arg z(x)} - h(x)| \leqslant \sqrt{z_{11}(x)\, z_{22}(x)}$: therefore

$$|\,|q| - \phi/\sqrt{z_{11} z_{22}}| \leqslant 1.$$

We have $v = -\arg(\phi/\sqrt{z_{11}z_{22}})$. Thus $|v| < \pi/2$ and $\sin|v| \leqslant 1/|q|$.

The sufficiency follows in a similar way. If there exist N such that $z(x) = |z(x)|\, e_N(x)$ then $v(x) = -\arg(e_{-N-n}(x)h(x))$. Applying the lemmas of [4] we obtain the characterization.

An answer for our problem is given by the next result, which is a generalization of the theorem of Helson and Sarason (see [9]) in a form similar to the extension given by Pourahmadi (see [17, 14]). This result characterizes only the bounding measure $\mu^{\#}$ and not the measure μ.

THEOREM 2.3. *Let* $a > 1$, $n \geqslant 0$, $r = (a-1)/(a+1)$, $\mu \in M_{q\times q}([0, 2\pi])$, *and* $\mu^{\#} \in M_{1\times 1}([0, 2\pi])$, $\mu \geqslant 0$, $\mu^{\#} \geqslant 0$. *If there exist* $c, d \in R$, $0 < c < d < \infty$ *such that* $c \cdot \mu^{\#} \cdot I_q \leqslant \mu \leqslant d \cdot \mu^{\#} \cdot I_q$, *then the following properties are equivalent*:

(a) $(\mu, \mu) \in R_{q\times q}([0, 2\pi], n, a)$. (b) $\rho_n(\mu) \leqslant r$.

(c) $(\mu^{\#}, \mu^{\#}) \in R_{1\times 1}([0, 2\pi], n, a)$. (d) $\rho_n(\mu^{\#}) \leqslant r$.

(e) $d\mu^{\#}(x) = w(x)\, dx$, *where* $w = |h(x)|\, e^{u(x)} = |P|^2 \exp(u + \tilde{v})$ *is a Helson–Sarason function of type* n *with* $h \in H^1([0, 2\pi])$ *and* $v(x) = -\arg(e_{-n}(x)h(x))$ *such that*

$$\|v\|_\infty \leqslant V(r) < \pi/2 \qquad \text{and} \qquad |u| \leqslant U(r, v).$$

Proof. From Propositions 1.2 and 2.1, we prove part of the result using

$$\mu_{\alpha\beta} = \frac{e_{n(\beta-\alpha)}(a + (-1)^{|\beta-\alpha|+1})\mu}{2\sqrt{a}} \qquad \text{for} \quad \alpha, \beta = 1, 2$$

$$\mu^{\#}_{\alpha\beta} = \frac{e_{n(\beta-\alpha)}(a + (-1)^{|\beta-\alpha|+1})\mu^{\#}}{2\sqrt{a}} \qquad \text{for} \quad \alpha, \beta = 1, 2.$$

The rest of theorem follows from the unidimensional theorem of Helson and Sarason in the form given by Arocena, Cotlar, and Sadosky, see [4].

The next result is also a matricial extension of the theorem of Helson and Sarason.

In the rest of the paper w and y will always be:

$$w = \begin{bmatrix} w_{11} & \cdots & w_{1q} \\ \vdots & \ddots & \vdots \\ w_{q1} & \cdots & w_{qq} \end{bmatrix}, \qquad y = \begin{bmatrix} y_{11} & \cdots & y_{1q} \\ \vdots & \ddots & \vdots \\ y_{q1} & \cdots & y_{qq} \end{bmatrix}$$

and $\mu = \mu_s + w(x)\, dx$, where μ_s is the singular part of μ with respect to Lebesgue measure.

THEOREM 2.4.(I) *Let* $a>1$, $n\geqslant 0$, $r=(a-1)/(a+1)$, $\mu\in M_{q\times q}([0, 2\pi])$, $\mu\geqslant 0$, *and* $\mu=\mu_s+w(x)\,dx$. *Let*

$$r_{kj}(x)=r\,\frac{\sqrt{w_{kk}(x)\,w_{jj}(x)}}{|w_{kj}(x)|}.$$

If $r_{kj}(x)<1$ *then the following conditions are equivalent*:

(a) $(\mu, \mu)\in R_{q\times q}([0, 2\pi], n, a)$.

(b) $\rho_n(\mu)\leqslant r$.

(c) $d\mu=w(x)\,dx$ *and for all* $k, j\in\{1, ..., q\}$ *there exist functions* $g_{kj}(x)=|h_{kj}(x)|\,e^{u_{kj}(x)}$, *where* $h_{kj}\in H^1([0, 2\pi])$ *such that*:

(i) $w_{kk}(x)=g_{kk}(x)$ *and it is a Helson–Sarason function of type n*.

(ii) $|w_{kj}(x)|=\sqrt{(1-r^2)(g_{kj}(x))^2+r^2 g_{kk}(x)\,g_{jj}(x)}$.

(iii) *Let* $v_{kj}(x)=-\arg(e_{-n}(x)h_{kj}(x)e^{-ix\cdot\arg w_{kj}(x)})$; *then*

$$|u_{kj}(x)|\leqslant U(r_{kj}(x), v_{kj}(x)) \qquad \text{and} \qquad |v_{kj}(x)|\leqslant V(r_{kj}(x))<\pi/2.$$

If $w_{kj}(x)=|w_{kj}(x)|\,e^{ixN_{kj}}$ *for a positive integer* N_{kj} *then* g_{kj} *is a Helson–Sarason function of type* $n+N_{kj}$.

(II) *Let* $a>1$, $n\geqslant 0$, $\mu, \nu\in M_{q\times q}([0, 2\pi])$, $\mu\geqslant 0$, $\nu\geqslant 0$, $d\mu=w(x)\,dx$, *and* $d\nu=y(x)\,dx$. *Let*

$$r_{a;kj}(x)=\frac{\sqrt{(ay_{kk}(x)-w_{kk}(x))(ay_{jj}(x)-w_{jj}(x))}}{|ay_{kj}(x)+w_{kj}(x)|}.$$

If $r_{a;kj}(x)<1$ *then the following properties are equivalent*:

(a_1) $(\mu, \nu)\in R_{q\times q}([0, 2\pi], n, a)$.

(b_1) *For all* $k, j\in\{1, ..., q\}$ *there exist functions* $g_{kj}(x)=|h_{kj}(x)|\,e^{u_{kj}(x)}$, *where* $h_{kj}\in H^1([0, 2\pi])$ *such that*:

(i) $\sqrt{y_{kk}(x)\,w_{kk}(x)}=g_{kk}(x)$ *and it is a Helson–Sarason function of type n*.

(ii) $|ay_{kj}(x)+w_{kj}(x)|$
$$=\sqrt{4a(g_{kj}(x))^2+(ay_{kk}(x)-w_{kk}(x))(ay_{jj}(x)-w_{jj}(x))}.$$

(iii) *Let* $v_{kj}(x)=-\arg(e_{-n}(x)h(x)\,e^{-ix\cdot\arg(ay_{kj}(x)+w_{kj}(x))})$; *then*

$$|u_{kj}(x)|\leqslant U(r_{a;kj}(x), v_{kj}(x))$$

and

$$|v_{kj}(x)|\leqslant V(r_{a;kj}(x))<\pi/2.$$

If $ay_{kj}(x)+w_{kj}(x)=|ay_{kj}(x)+w_{kj}(x)|\, e^{ixN_{kj}}$ *for a positive integer* N_{kj} *then* g_{kj} *is a Helson–Sarason function of type* $n+N_{kj}$.

Proof. We shall only prove part (II), part (I) follows in a similar way. From Propositions 2.1 and 1.1, we have that (a_1) is equivalent to the statement that the 2×2 matrix,

$$\left(\frac{e_{n(\beta-\alpha)}(x)(ay_{k_\alpha k_\beta}(x)+(-1)^{|\beta-\alpha|+1}\, w_{k_\alpha k_\beta}(x))\, dx}{2\sqrt{a}}\right)_{\alpha,\beta=1,2}$$

is weakly positive for every $k_1, k_2\in\{1,\dots,q\}$. By Proposition 2.2 for every $k, j\in\{1,\dots,q\}$ there exist functions $g_{kj}(x)=|h_{kj}(x)|\, e^{u_{kj}(x)}$ with $h_{kj}\in H^1([0,2\pi])$ such that

$$\sqrt{\left(\frac{ay_{kk}(x)-w_{kk}(x)}{2\sqrt{a}}\right)\left(\frac{ay_{jj}(x)-w_{jj}(x)}{2\sqrt{a}}\right)}=\frac{1}{\sqrt{|q_{kj}(x)|^2-1}}\, g_{kj}(x)$$

and

$$\frac{|ay_{kj}(x)+w_{kj}(x)|}{2\sqrt{a}}=|q_{kj}(x)|\,\frac{1}{\sqrt{|q_{kj}(x)|^2-1}}\, g_{kj}(x),$$

where

$$q_{kj}(x)=\frac{(ay_{kj}(x)+w_{kj}(x))\, e_n(x)}{\sqrt{(ay_{kk}(x)-w_{kk}(x))(ay_{jj}(x)-w_{jj}(x))}}$$

and $v_{kj}(x)=-\arg(e_{-n}(x)\, h_{kj}(x)\, e^{-ix\cdot\arg(ay_{kj}(x)+w_{kj}(x))})$ satisfy the inequalities

$$|u_{kj}(x)|\leqslant U\left(\frac{1}{|q_{kj}(x)|};\, v_{kj}(x)\right)\qquad\text{and}\qquad |v_{kj}(x)|\leqslant V\left(\frac{1}{|q_{kj}(x)|}\right)<\frac{\pi}{2}.$$

If $k=j$, then

$$|q_{kk}(x)|=\frac{ay_{kk}(x)+w_{kk}(x)}{ay_{kk}(x)-w_{kk}(x)}=\frac{1}{r_{a;kk}(x)}$$

Thus

$$\sqrt{y_{kk}(x)\, w_{kk}(x)}=\sqrt{|q_{kk}(x)|^2-1}\left(\frac{ay_{kk}(x)-w_{kk}(x)}{2\sqrt{a}}\right)=g_{kk}(x)$$

and

$$\frac{|q_{kk}(x)|}{\sqrt{|q_{kk}(x)|^2-1}}=\frac{1}{\sqrt{1-r_{a;kk^2}(x)}}.$$

So we have the first part.

For every k, j,

$$|q_{kj}(x)| = \frac{|ay_{kj}(x) + w_{kj}(x)|}{\sqrt{(ay_{kk}(x) - w_{kk}(x))(ay_{jj}(x) - w_{jj}(x))}}$$

and so $g_{kj} = (1/2\sqrt{a})\sqrt{|ay_{kj} + w_{kj}|^2 - (ay_{kk} - w_{kk})(ay_{jj} - w_{jj})}$. Therefore, $|ay_{kj} + w_{kj}| = \sqrt{4a(g_{kj})^2 + (ay_{kk} - w_{kk})(ay_{jj} - w_{jj})}$, and the proof that a_1 implies b_1 is finished. That b_1 implies a_1 is similarly a consequence of Propositions 1.1, 2.1, and 2.2.

The process $(X_n)_{n \in Z}$ is said to be linearly completely regular if $\lim_{n \to \infty} \rho_n(\mu) = 0$.

The following theorem characterizes some q-variate linearly completely regular discrete processes and generalizes another theorem of Helson and Sarason (see [9]).

For any $\mu \in M_{q \times q}([0, 2\pi])$, positive $\mu = \mu_s + w(x)\,dx$, let

$$M_\mu = \max_{k;j;x} \left\{ \frac{\sqrt{w_{kk}(x)\, w_{jj}(x)}}{|w_{kj}(x)|} \right\} \qquad \text{and} \qquad m_\mu = \min_{k;j;x} \left\{ \frac{\sqrt{w_{kk}(x)\, w_{jj}(x)}}{|w_{kj}(x)|} \right\}.$$

THEOREM 2.5. *Let* $\mu \in M_{q \times q}([0, 2\pi])$, $\mu \geqslant 0$, $\mu = \mu_s + w(x)\,dx$. *If* $0 < m_\mu \leqslant M_\mu < \infty$, *then the following properties are equivalent*:

(a) $\lim_{n \to \infty} \rho_n(\mu) = 0$

(b) *For every* $\varepsilon > 0$, *there exist* $n > 0$ *and* $r_\varepsilon < 1/M_\mu$ *such that for every* $r < r_\varepsilon$, *for* $k, j \in \{1, ..., q\}$ *there exist functions* $g_{\varepsilon kj} = |h_{\varepsilon kj}|\, e^{u_{\varepsilon kj}}$, *where* $h_{\varepsilon kj} \in H^1([0, 2\pi])$ *that satisfy* $w_{kk}(x) = g_{\varepsilon kk}(x)$ *and it is a Helson–Sarason function of type n*,

$$|w_{kj}(x)| = \sqrt{(1 - r^2)(g_{\varepsilon kj}(x))^2 + r^2 g_{\varepsilon kk}(x)\, g_{\varepsilon jj}(x)}$$

and

$$\|u_{\varepsilon kj}\|_\infty + \|v_{\varepsilon kj}\|_\infty < \varepsilon,$$

where $v_{\varepsilon kj}(x) = -\arg(e_{-n}(x) h_{\varepsilon kj}(x)\, e^{-ix \cdot \arg w_{kj}(x)})$.

In this case we also have that $w_{kj}(x) = \lim_{\varepsilon \to 0} g_{\varepsilon kj}(x)$. If $w_{kj}(x) = |w_{kj}(x)|\, e^{ixN_{kj}}$ for a positive integer N_{kj} then g_{kj} is a Helson–Sarason function of type $n + N_{kj}$.

Proof. $(\to)$ Given $\varepsilon > 0$, there exists $r_\varepsilon \in (0, 1)$ such that for all $r < r_\varepsilon$

$$\operatorname{arc\,cosh}\left(\frac{1}{\sqrt{1 - r^2 M_\mu^2}}\right) < \frac{\varepsilon}{2}, \qquad \operatorname{arc\,sin}(rM_\mu) < \frac{\varepsilon}{2}, \qquad r < \frac{1}{M_\mu}.$$

Let

$$r_{kj}(x) = r\,\frac{\sqrt{w_{kk}(x)\,w_{kjj}(x)}}{|w_{kj}(x)|}$$

then $r_{kj}(x) < r \cdot M_\mu < 1$.

By hypothesis, there exists n such that $\rho_n(\mu) \leqslant r$. From Theorem 2.4, for every $k, j \in \{1, ..., q\}$, w_{kj} has the required representation and

$$|u_{\varepsilon kj}(x)| \leqslant \operatorname{arc\,cosh}\left(\frac{\cos v_{\varepsilon kj}(x)}{\sqrt{1 - r_{kj}^2(x)}}\right),$$

$$|v_{\varepsilon kj}(x)| \leqslant \arcsin r_{kj}(x) < \frac{\pi}{2}.$$

Then

$$|u_{\varepsilon kj}(x)| \leqslant \operatorname{arc\,cosh}\left(\frac{1}{\sqrt{1 - r^2 M_\mu^2}}\right) < \frac{\varepsilon}{2}$$

$$|v_{\varepsilon kj}(x)| \leqslant \arcsin (rM_\mu) < \frac{\varepsilon}{2}.$$

$(\leftarrow)$ For $r > 0$, let

$$r_{kj}(x) = r\,\frac{\sqrt{w_{kk}(x)\,w_{jj}(x)}}{|w_{kj}(x)|},$$

then $rm_\mu < r_{kj}(x) < rM_\mu$. As

$$\lim_{x \to 0} \operatorname{arc\,cosh}\left(\frac{\cos x}{\sqrt{1 - r^2 m_\mu^2}}\right) = L > 0,$$

there is $\delta > 0$ such that

$$\operatorname{arc\,cosh}\left(\frac{\cos x}{\sqrt{1 - r^2 m_\mu^2}}\right) > \frac{L}{2} > 0 \qquad \text{if} \quad |x| \leqslant \delta.$$

Let $\varepsilon = \min\{\delta; L/2\}$ then

$$\varepsilon \leqslant \frac{L}{2} < \operatorname{arc\,cosh}\left(\frac{\cos \varepsilon}{\sqrt{1 - r^2 m_\mu^2}}\right).$$

And $\varepsilon < \arcsin (rm_\mu)$, (when defining arc cosh we take $\cos \varepsilon > \sqrt{1 - r^2 m_\mu^2}$).

There is $n > 0$ and $r_\varepsilon < 1/M_\mu$ such that for every $r < r_\varepsilon$ and $k, j \in \{1, ..., q\}$, w_{kj} has the representation and $\|u_{\varepsilon kj}\|_\infty + \|v_{\varepsilon kj}\|_\infty < \varepsilon$. Therefore $\|v_{\varepsilon kj}\|_\infty < \varepsilon < \arcsin(rm_\mu) < \arcsin r_{kj}(x)$ and

$$|u_{\varepsilon kj}| < \varepsilon < \operatorname{arc\,cosh}\left(\frac{\cos \varepsilon}{\sqrt{1-r^2 m_\mu^2}}\right)$$
$$< \operatorname{arc\,cosh}\left(\frac{\cos v_\varepsilon}{\sqrt{1-(r_{kj}(x))^2}}\right).$$

By Theorem 2.4, $\rho_n(\mu) \leqslant r$. The result follows because $\rho_n(\mu)$ decreases.

The following result was announced in [5] for the continuous unidimensional case.

THEOREM 2.6. *Let* $\{r_n\}_{n\geqslant 0} \subset (0,1)$ *with* $\lim_{n\to\infty} r_n = 0$, $\mu \in M_{q\times q}([0, 2\pi])$, $\mu \geqslant 0$, $\mu = \mu_s + w(x)\,dx$, *if* $0 < m_\mu \leqslant M_\mu < \infty$. *Then the following properties are equivalent*:

(a) $\rho_n(\mu) = O(r_n)$, $n \to \infty$.

(b) *There exist* s *and* $n_0 \geqslant 0$ *such that for every* $n \geqslant n_0$, *for* $k, j \in \{1, ..., q\}$ *there exist functions* $g_{nkj} = |h_{nkj}(x)|\, e^{u_{nkj}(x)}$, *where* $h_{nkj} \in H^1([0, 2\pi])$ *such that*:

$$w_{kk}(x) = g_{nkk}(x) \text{ and it is a Helson–Sarason function of type } n,$$
$$|w_{kj}(x)| = \sqrt{(1-(sr_n)^2)(g_{nkj}(x))^2 + (sr_n)^2\, g_{nkk}(x)\, g_{njj}(x)},$$
$$\|v_{nkj}\|_\infty = O(r_n) \qquad \text{and} \qquad \frac{\cosh|u_{nkj}|}{\cos v_{nkj}} - 1 = O(r_n^2), \qquad n\to\infty,$$

where $v_{nkj}(x) = -\arg(e_{-n}(x)\, h_{nkj}(x)\, e^{-ix\cdot \arg w_{kj}(x)})$.

In this case we also have $w_{kj}(x) = \lim_{n\to\infty} g_{nkj}(x)$.

If $w_{kj}(x) = |w_{kj}(x)|\, e^{ixN_{kj}}$ *for a positive integer* N_{kj} *then* g_{kj} *is a Helson–Sarason function of type* $n + N_{kj}$.

Proof. Let

$$r_{nkj}(x) = r_n \frac{\sqrt{w_{kk}(x)\, w_{jj}(x)}}{|w_{kj}(x)|},$$

then $r_n m_\mu \leqslant r_{nkj}(x) \leqslant r_n M_\mu$ a.e.

(a) $\to$ (b) There exist $c, \delta > 0$ such that if $0 < x < \delta$, then $\arcsin x < cx$. If $0 < x^2 < 1/2$ then $1/\sqrt{1-x^2} \leqslant 1 + x^2$.

There exist s and n_1 such that if $n \geqslant n_1$, then $\rho_n(\mu) \leqslant sr_n < 1/M_\mu$, thus $sr_{nkj}(x) < 1$. By Theorem 2.4 for every $k, j \in \{1, ..., q\}$ w_{kj} has the required representation, with

$$|u_{nkj}(x)| \leqslant \operatorname{arc\,cosh}\left(\frac{\cos v_{nkj}(x)}{\sqrt{1-(sr_{nkj}(x))^2}}\right),$$

$$|v_{nkj}(x)| \leqslant \arcsin(sr_{nkj}(x)) < \frac{\pi}{2}.$$

Thus

$$|u_{nkj}(x)| \leqslant \operatorname{arc\,cosh}\left(\frac{\cos v_{nkj}(x)}{\sqrt{1-s^2r_n^2M_\mu^2}}\right)$$

$$|v_{nkj}(x)| \leqslant \arcsin(sr_n M_\mu) < \frac{\pi}{2}.$$

But there is n_2 such that if $n \geqslant n_2$ then $sr_n M_\mu < \delta$ and $sr_n M_\mu < 1/\sqrt{2}$. Let $n_0 = \max\{n_1, n_2\}$ then for each $n \geqslant n_0$,

$$\frac{\cosh|u_{nkj}|}{\cos v_{nkj}} \leqslant \frac{1}{\sqrt{1-s^2r_n^2M_\mu^2}}$$
$$\leqslant 1+s^2r_n^2M_\mu^2, \qquad |v_{nkj}(x)| \leqslant csr_n M_\mu.$$

(b) $\rightarrow$ (a) For every $x \in R$, $1+x^2 \leqslant 1/\sqrt{1-2x^2}$. There exist $\lambda > 0$, $\delta \in (0, 1)$ such that if $0 < x < \delta$, then $\lambda x < \arcsin x$.

There exist c, s, and n_0 such that if $n \geqslant n_0$, for every $k, j \in \{1, ..., q\}$, w_{kj} has the representation expressed in terms of s,

$$\|v_{nkj}\|_\infty \leqslant cr_n, \qquad \frac{\cosh|u_{nkj}|}{\cos v_{nkj}} - 1 \leqslant c^2r_n^2.$$

As $cr_n \rightarrow 0$, we can take c such that $cM_\mu/m_\mu\lambda < 1$ and $\sqrt{2}\, cM_\mu/m_\mu < 1$.

There exist n_1 such that if $n \geqslant n_1$, $cr_n/\lambda < \delta$.

Let $N = \max\{n_0, n_1\}$; then for every $n \geqslant N$,

$$|v_{nkj}(x)| \leqslant cr_n \leqslant \arcsin((c/\lambda)\, r_n) \leqslant \arcsin((c/m_\mu\lambda)\, r_{nkj}(x)). \qquad \text{And}$$

$$\cosh|u_{nkj}(x)| \leqslant (1+c^2r_n^2)\cos v_{nkj}(x)$$
$$\leqslant \frac{\cos v_{nkj}(x)}{\sqrt{1-2c^2(r_{nkj}(x)/m_\mu)^2}}.$$

Let $d = \max\{c/m_\mu\lambda; \sqrt{2}\, c/m_\mu\}$; then $dr_{nkj}(x) < 1$. So

$$|v_{nkj}(x)| \leqslant \arcsin(dr_{nkj}(x)) \qquad \text{and}$$

$$|u_{nkj}(x)| \leqslant \operatorname{arc\,cosh}\left(\frac{\cos v_{nkj}(x)}{\sqrt{1-d^2(r_{nkj}(x))^2}}\right).$$

By Theorem 2.4, for each $n \geqslant n_0$, $\rho_n(\mu) \leqslant dr_n$.

3. Extension of the Helson–Sarason Theorem in the Continuous Case

All done before for positive and weakly positive measures in $[0, 2\pi]$ can be done in R, considering instead of the spaces $P(C_{q\times q})$, $P_1(C_{q\times q})$, $P_2(C_{q\times q})$, and $P_{(n)}(C_{q\times q})$ the following spaces:

$$E(C_{q\times q}) = \begin{cases} \phi\colon R \to C_{q\times q};\ \phi(x) = \sum_t e_t(x)\,\hat{\phi}(t) \text{ and} \\ \hat{\phi}\colon R \to C_{q\times q} \text{ is a function of finite support} \end{cases}$$

$$E_1(C_{q\times q}) = \{\phi \in E(C_{q\times q});\ \hat{\phi}(t) = 0 \text{ for every } t < 0\},$$

$$E_2(C_{q\times q}) = \{\phi \in E(C_{q\times q});\ \hat{\phi}(t) = 0 \text{ for every } t \geqslant 0\}, \text{ and}$$

$$E_{(t)}(C_{q\times q}) = \{\phi = e_t\phi_1 + \phi_2 \text{ for } \phi_1 \in E_1(C_{q\times q});\ \phi_2 \in E_2(C_{q\times q})\}, \text{ for } t \geqslant 0.$$

In this paper we shall also use the following notation:

$$I_R\colon R \to R, \qquad I_R(x) = x,$$

$$E_{1,b}(C_{q\times q}) = \left\{\frac{\phi}{(I_R + i)^{b/2}};\ \phi \in E_1(M_q)\right\},$$

$$E_{2,b}(C_{q\times q}) = \left\{\frac{\phi}{(I_R - i)^{b/2}};\ \phi \in E_2(M_q)\right\}$$

and for $t \geqslant 0$,

$$E_{(t),b}(C_{q\times q}) = \left\{f = \frac{e_t\phi_1}{(I_R + i)^{b/2}} + \frac{\phi_2}{(I_R - i)^{b/2}} \right.$$
$$\left. \text{for } \phi_1 \in E_1(C_{q\times q});\ \phi_2 \in E_2(C_{q\times q})\right\}.$$

For $C_{1\times 1}$ these spaces have been used in [2, 6, 7].

A measure μ in R, with values in $C_{q\times q}$ is tempered of order less than or equal to b if $\mu/(x^2+1)^{b/2}$ is a finite measure. Let

$$M^b_{q\times q}(R) = \{C_{q\times q}\text{-valued tempered Borel measures } \mu \text{ in } R \text{ of order less than or equal to } b\}.$$

If $\mu \in M^b_{q\times q}(R)$ then $E_{(t),b}(C_{q\times q}) \subset L^2_q(R, \mu)$ for all $t \geqslant 0$.

Let $\mu, \nu \in M^b_{q\times q}(R)$ be positive. Let $\mathbf{H}$ be the Hilbert transform defined in $E_{(t),b}(C_{q\times q})$ by

$$\mathbf{H}(f) = -ie_t\frac{\phi_1}{(I_R + i)^{b/2}} + i\frac{\phi_2}{(I_R - i)^{b/2}}$$

if

$$f = e_t \frac{\phi_1}{(I_R + i)^{b/2}} + \frac{\phi_2}{(I_R - i)^{b/2}}$$

with $\phi_1 \in E_1(C_{q\times q})$ and $\phi_2 \in E_2(C_{q\times q})$.

For $a \geqslant 1$, let

$$\begin{aligned} R^b_{q\times q}(R, t, a) = \{(\mu; \nu) \in M^b_{q\times q}(R) \times M^b_{q\times q}(R); \mu \geqslant 0;\ \nu \geqslant 0;\\ \text{such that } \operatorname{tr} \textstyle\int_{-\infty}^{\infty} (\mathbf{H}f)\, d\mu (\mathbf{H}f)^* \leqslant a \operatorname{tr} \int_{-\infty}^{\infty} f\, d\nu\, f^*\\ \text{for all } f \in E_{(t);b}(C_{q\times q})\}. \end{aligned}$$

We consider the problem of characterizing the elements of $R^b_{q\times q}(R, t, a)$.

Let (Ω, A, P) be a probability space. Let $(X_t)_{t\in R}$ be a q-variate weakly stationary process, that is, $X_t = (X_{t(1)}, ..., X_{t(q)})$ $X_{t(j)} \in L^2(\Omega, A, P)$ for $j = 1, ..., q$ and the covariance depends only of $t - s$: $\Gamma_{t-s} = E(X_{t(i)} \overline{X_{s(j)}}))_{i,j=1,...,q}$. The Bochner theorem assures the existence of the spectral measure of the process $\mu \in M^0_{q\times q}(R)$, which is positive hermitian. The maximal correlation coefficient of the process $(X_t)_{t\in R}$ is

$$\rho_t(\mu) = \sup_{\substack{f_1 \in E_1(C_{q\times q});\, f_2 \in E_2(C_{q\times q})\\ \langle\langle f_1; f_1\rangle\rangle_\mu = \langle\langle f_2; f_2\rangle\rangle_\mu = 1}} |\langle\langle e_t f_1; f_2\rangle\rangle_\mu|.$$

Let $\mu \in M^b_{q\times q}(R)$, positive. Let $\rho_{t;b}(\mu)$ the cosine of the angle between $e_t E_{1;b}(C_{q\times q})$ and $E_{2;b}(C_{q\times q})$,

$$\rho_{t;b}(\mu) = \sup_{\substack{f_1 \in E_{1;b}(C_{q\times q});\, f_2 \in E_{2;b}(C_{q\times q})\\ \langle\langle f_1; f_1\rangle\rangle_\mu = \langle\langle f_2; f_2\rangle\rangle_\mu = 1}} |\langle\langle e_t f_1; f_2\rangle\rangle_\mu|.$$

PROPOSITION 3.1. *Let $a > 1$ and $t \geqslant 0$. Let $\mu, \nu \in M^b_{q\times q}(R)$, $\mu \geqslant 0$, $\nu \geqslant 0$, and, for $\alpha, \beta = 1, 2$ let*

$$\mu_{\alpha\beta} = \frac{e_{t(\beta-\alpha)}(a\nu + (-1)^{|\beta-\alpha|+1}\mu)}{2\sqrt{a}\,(I_R + (-1)^{\alpha-1} i)^{b/2}\,(I_R + (-1)^{\beta} i)^{b/2}}.$$

Then the following properties are equivalent:

(a) $(\mu, \nu) \in R^b_{q\times q}(R, t, a)$.

(b) $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ *is weakly positive.*

If $\mu = \nu$ *and* $r = (a-1)/(a+1)$ *then*

$$\mu_{\alpha\beta} = \frac{e_{t(\beta-\alpha)} r^{1-|\beta-\alpha|}\,\mu}{\sqrt{1-r^2}\,(I_R+(-1)^{\alpha-1}\,i)^{b/2}\,(I_R+(-1)^{\beta}\,i)^{b/2}}$$

for $\alpha, \beta = 1, 2$ *and conditions* (a) *and* (b) *are also equivalent to*

(c) $\rho_{t,b}(\mu) \leqslant r.$

In this case μ *is absolutely continuous.*

PROPOSITION 3.2. *Let* $(z_{\alpha\beta})_{\alpha,\beta=1,2}$, *where* $z_{\alpha\beta}/(x^2+1)^{b/2} \in L^1(R)$ *be such that*

(i) $z_{11}(x) > 0$ *a.e.*, $z_{22}(x) > 0$ *a.e.*, *and* $z_{12}(x) = \overline{z_{21}(x)}$ *a.e.*

(ii) $0 < \sqrt{z_{11}(x)\, z_{22}(x)} < |z_{12}(x)|$ *a.e.*

Let t *be a nonnegative real number; we shall call* $z(x) = z_{12}(x)\, e_{-t}(x)$ *and* $q = z_{12}/\sqrt{z_{11} z_{22}}$. *The following conditions are equivalent*:

(a) $$Z_b = \left(\frac{z_{\alpha\beta}(x)\, dx}{(x+(-1)^{\alpha-1}\, i)^{b/2}\,(x+(-1)^{\beta}\, i)^{b/2}}\right)_{\alpha,b=1,2}$$

is weakly positive.

(b) $$\frac{|z_{12}(x)|}{|q(x)|} = \sqrt{z_{11}(x) z_{22}(x)} = \frac{(x^2+1)^{b/2}}{\sqrt{|q(x)|^2-1}}\,|h(x)|\, e^{u(x)},$$

where $h \in H^1(R)$ *and* $v(x) = -\arg((x+i)^b\, e_{-t}(x) h(x)\, e^{-ix\cdot \arg z(x)})$ *satisfy*

$$|u(x)| \leqslant U\left(\frac{1}{|q(x)|}; v(x)\right) \qquad \text{and} \qquad |v(x)| \leqslant V\left(\frac{1}{|q(x)|}\right).$$

Proof. If Z_b is weakly positive, then from the lifting theorem there exists $h \in H^1(R)$ and $\phi(x) = (x+i)^b\, e_{-t}(x) h(x)\, e^{-ix\cdot \arg z(x)}$ such that

$$0 \geqslant \frac{|z_{12}|^2}{(x^2+1)^b} - 2\,\mathrm{Re}\left(\frac{h\overline{z_{12}}}{(x-i)^b}\right) + |h|^2 - \frac{z_{11} z_{22}}{(x^2+1)^b} \qquad \text{a.e.}$$

Then $0 \geqslant (|q|^2-1)\, z_{11} z_{22} - 2|q|(\mathrm{Re}\,\phi)\sqrt{z_{11} z_{22}} + |\phi|^2$ a.e. If we call $v = -\arg\phi$ and

$$u(x) = \log\left(\frac{\sqrt{z_{11} z_{22}}\,\sqrt{|q|^2-1}}{|\phi|}\right),$$

then

$$|u(x)| \leqslant \text{arc cosh}\left(\frac{|q| \cos v(x)}{\sqrt{|q|^2-1}}\right)$$

and

$$\sqrt{z_{11}(x)\, z_{22}(x)} = \frac{(x^2+1)^{b/2}}{\sqrt{|q(x)|^2-1}}\, |h(x)|\, e^{u(x)}.$$

And

$$\left|\frac{|z_{12}|\, e_t}{(x+i)^b} - h\right| \leqslant \frac{\sqrt{z_{11} z_{22}}}{(x^2+1)^{b/2}} \qquad \text{so} \qquad \left||q| - \frac{(x+i)^b\, e_{-t} h}{\sqrt{z_{11} z_{22}}}\right| \leqslant 1.$$

The proof is finished in the same way as in the discrete case.

A first answer to the stated problem is given by the next result, which is an extension to R of the generalizations given before of the theorem of Pourahmadi cf. [17].

THEOREM 3.3. *Let* $a>1$, $t \geqslant 0$, $r=(a-1)/(a+1)$; $\mu \in M^b_{q\times q}(R)$, *and* $\mu^\# \in M^b_{1\times 1}(R)$, $\mu \geqslant 0$, $\mu^\# \geqslant 0$. *If there exist* $c, d \in R$, $0<c<d<\infty$ *such that* $c \cdot \mu^\# \cdot I_q \leqslant \mu \leqslant d \cdot \mu^\# \cdot I_q$ *then the following properties are equivalent*:

(a) $(\mu, \mu) \in R^b_{q\times q}(R, t, a)$. (b) $\rho_{t,b}(\mu) \leqslant r$.

(c) $(\mu^\#, \mu^\#) \in R^b_{1\times 1}(R, t, a)$. (d) $\rho_{t,b}(\mu^\#) \leqslant r$.

(e) $d\mu^\#(x) = (x^2+1)^{b/2}\, |h(x)|\, e^{u(x)}\, dx$, *where* $h \in H^1(R)$ *and* $v(x) = -\arg((x+i)^b\, e_{-t}(x) h(x))$ *satisfy*

$$\|v\|_\infty \leqslant V(r) < \pi/2 \qquad \text{and} \qquad |u| \leqslant U(r, v).$$

We give a constructive characterization of the exponential Helson–Szegö–Sarason type, for the continuous case.

THEOREM 3.4. (I) *Let* $a>1$, $t \geqslant 0$, $r=(a-1)/(a+1)$, $\mu \in M^b_{q\times q}(R)$, $\mu \geqslant 0$, *and* $\mu = \mu_s + w(x)\, dx$. *Let*

$$r_{kj}(x) = r\, \frac{\sqrt{w_{kk}(x)\, w_{jj}(x)}}{|w_{kj}(x)|}$$

if $r_{kj}(x) < 1$ *then the following conditions are equivalent*:

(a) $(\mu, \mu) \in R^b_{q\times q}(R, t, a)$.

(b) $\rho_{t,b}(\mu) \leqslant r$.

(c) *$d\mu = w(x)\,dx$ and for all $k, j \in \{1, ..., q\}$ there exist functions $g_{kj}(x) = (x^2+1)^{b/2}\,|h_{kj}(x)|\,e^{u_{kj}(x)}$ where $h_{kj} \in H^1(R)$ such that:*

(i) $w_{kk}(x) = g_{kk}(x)$.

(ii) $|w_{kj}(x)| = \sqrt{(1-r^2)(g_{kj}(x))^2 + r^2 g_{kk}(x)\,g_{jj}(x)}$.

(iii) *Let* $v_{kj}(x) = -\arg((x+i)^b\,e_{-t}(x)\,h_{kj}(x)\,e^{-ix\cdot\arg w_{kj}(x)})$; *then*

$$|u_{kj}(x)| \leqslant U(r_{kj}(x), v_{kj}(x))$$

and

$$|v_{kj}(x)| \leqslant V(r_{kj}(x)) \leqslant \pi/2.$$

(II) *Let $a > 1$, $t \geqslant 0$, and $\mu, \nu \in M^b_{q\times q}(R)$ positive, $d\mu = w(x)\,dx$, and $d\nu = y(x)\,dx$. Let*

$$r_{a,kj}(x) = \frac{\sqrt{(ay_{kk}(x) - w_{kk}(x))(ay_{jj}(x) - w_{jj}(x))}}{|ay_{kj}(x) + w_{kj}(x)|},$$

if $r_{a;kj}(x) < 1$; then the following properties are equivalent:

(a_1) $(\mu, \nu) \in R^b_{q\times q}(R, t, a)$.

(b_1) *For every $k, j \in \{1, ..., q\}$ there exist functions g_{kj}, $g_{kj}(x) = (x^2+1)^{b/2}\,|h_{kj}(x)|\,e^{u_{kj}(x)}$, where $h_{kj} \in H^1(R)$ such that*

(i) $\sqrt{y_{kk}(x)\,w_{kk}(x)} = g_{kk}(x)$.

(ii) $|ay_{kj}(x) + w_{kj}(x)|$
$$= \sqrt{4a(g_{kj}(x))^2 + (ay_{kk}(x) - w_{kk}(x))(ay_{jj}(x) - w_{jj}(x))}.$$

(iii) *Let* $v_{kj}(x) = -\arg((x+i)^b\,e_{-t}(x)\,h_{kj}(x)\,e^{-ix\cdot\arg(ay_{kj}(x)+w_{kj}(x))})$;
then

$$|u_{kj}(x)| \leqslant U(r_{a,kj}(x), v_{kj}(x))$$

and

$$|v_{kj}(x)| \leqslant V(r_{a,kj}(x)) < \pi/2.$$

Proof. From Proposition 3.1 and a result similar to Proposition 1.1, but for R, it follows that (a_1) is equivalent to

$$\left(\frac{e_{t(\beta-\alpha)}(x)(ay_{k_\alpha k_\beta}(x) + (-1)^{|\beta-\alpha|+1}\,w_{k_\alpha k_\beta}(x))\,dx}{2\sqrt{a}\,(x+(-1)^{\alpha-1}\,i)^{b/2}\,(x+(-1)^{\beta}\,i)^{b/2}}\right)_{\alpha,\beta=1,2}$$

is weakly positive for every $k_1 k_2 \in \{1, ..., q\}$.

So, arguing as in the proof of Theorem 2.4, part (II) of the theorem follows from Proposition 3.2. The proof of part (I) is similar.

For any $\mu \in M^b_{q\times q}(R)$ positive, let M_μ and m_μ be as before.

THEOREM 3.5. *Let $\mu \in M^b_{q\times q}(R)$, $\mu \geqslant 0$, $\mu = \mu_s + w(x)\,dx$, if $0 < m_\mu \leqslant M_\mu < \infty$. Then the following properties are equivalent*:

(a) $\lim_{t\to\infty} \rho_{t;b}(\mu) = 0$.

(b) *For every $\varepsilon > 0$, there exist $t > 0$ and $r_\varepsilon < 1/M_\mu$ such that for all $r < r_\varepsilon$, for $k, j \in \{1, ..., q\}$ there exist functions $g_{\varepsilon kj}(x) = (x^2+1)^{b/2}\,|h_{\varepsilon kj}(x)|\, e^{u_{\varepsilon kj}(x)}$, where $h_{\varepsilon kj} \in H^1(R)$ that satisfy*:

$$w_{kk}(x) = g_{\varepsilon kk}(x),$$
$$|w_{kj}(x)| = \sqrt{(1-r^2)(g_{\varepsilon kj}(x))^2 + r^2 g_{\varepsilon kk}(x)\, g_{\varepsilon jj}(x)},$$

and

$$\|u_{\varepsilon kj}\|_\infty + \|v_{\varepsilon kj}\|_\infty < \varepsilon,$$

where $v_{\varepsilon kj}(x) = -\arg((x+i)^b\, e_{-t}(x)\, h_{\varepsilon kj}(x)\, e^{-ix\cdot \arg w_{kj}(x)})$. In this case we also have that $w_{kj}(x) = \lim_{\varepsilon\to 0} g_{\varepsilon kj}(x)$.

This theorem follows similarly to Theorem 2.5 and characterizes some q-variate linearly completely regular continuous processes, that is, the processes with $\lim_{t\to\infty} \rho_t(\mu) = 0$. The last and next theorem generalizes the unidimensional results announced in [5].

THEOREM 3.6. *Let $\{r_t\}_{t\geqslant 0} \subset (0,1)$ with $\lim_{t\to\infty} r_t = 0$. Let $\mu \in M^b_{q\times q}(R)$, $\mu \geqslant 0$, $\mu = \mu_s + w(x)\,dx$, and if $0 < m_\mu \leqslant M_\mu < \infty$. Then the following are equivalent*:

(a) $\rho_{t,b}(\mu) = 0(r_t)$, $t \to \infty$.

(b) *There exist s and $t_0 \geqslant 0$ such that for all $t \geqslant t_0$, for $k, j \in \{1, ..., q\}$ there exist functions $g_{tkj}(x) = (x^2+1)^{b/2}\,|h_{tkj}(x)|\, e^{u_{tkj}(x)}$, $h_{tkj} \in H^1(R)$, such that*

$$w_{kk}(x) = g_{tkk}(x)$$
$$|w_{kj}(x)| = \sqrt{(1-(sr_t)^2)\,(g_{tkj}(x))^2 + (sr_t)^2\, g_{tkk}(x)\, g_{tjj}(x)},$$

$$\|v_{tkj}\|_\infty = 0(r_t),\ t\to\infty \qquad \textit{and} \qquad \frac{\cosh|u_{tkj}|}{\cos v_{tkj}} - 1 = O(r_t^2),\ t \to \infty,$$

where $v_{tkj}(x) = -\arg((x+i)^b\, e_{-t}(x) h_{tkj}(x)\, e^{-ix\cdot\arg w_{kj}(x)})$. In this case we also have $w_{kj}(x) = \lim_{t\to\infty} g_{tkj}(x) = 0$.

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